Linear Transformations of Vector Spaces

1. Definition:

- A linear transformation $T: V \to V'$ is a function that satisifies:

  1. Closure Under Addition
     $T(u+v) = T(u) + T(v)$

  2. Closure Under Scalar Multiplication
     $T(ru) = rT(u)$

for all vectors u and v in V and for all scalars r in R.

- The domain of T is the set V.

- The co-domain of T is the set V'.

- The image, or range, is the column space.

- The kernel is the nullspace.

- The inverse image is the set of vectors in V that map to V'. It is the inverse of the matrix representation.

- Let V, V', V'' be vector spaces.
  Let $T: V \to V'$ and $T': V' \to V''$ be linear transformations.
  Then, the composite function $T' \circ T: V \to V''$ is defined by $(T' \circ T)(v) = T'(T(v))$.

  Note, $T' \circ T$ is a linear transformation.

E.g. 1 Let F be the vector space of all functions $f: R \to R$ and let c be in R. Show that the evaluation function $T: F \to R$ defined by $T(f) = f(c)$ is a linear transformation.

Solution:

$$\begin{aligned} T(f+g) &= (f+g)(c) \\ &= f(c) + g(c) \\ &= T(f) + T(g) \end{aligned}$$

$$\begin{aligned} T(rf) &= (rf)(c) \\ &= rf(c) \\ &= rT(f) \end{aligned}$$

$\therefore$ $T: F \to R$ is a linear transformation

2. Properties of Linear Transformations:

Let V and V' be vector spaces and let $T: V \to V'$ be a linear transformation. Then,

1. $T(0) = 0'$
2. $T(V_1 - V_2) = T(V_1) - T(V_2)$
   for any vectors V_1 and V_2 in V

3. Bases and Linear Transformations:

Let $T: V \to V'$ be a linear transformation. Let B be a basis for V. For any vector v in V, the vector $T(v)$ is uniquely determined by the vectors $T(b)$ for all $b \in B$. In other words, if 2 linear transformations have the same value at each basis vector $b \in B$, then the 2 linear transformations are the same.

Proof:

Let T and T' be 2 linear transformations s.t. $T(b_i) = T'(b_i)$ for each vector b_i in B. Let $v \in V$. Then, there exists vectors $b_1, b_2, \ldots b_k$ in B and scalars $r_1, r_2, \ldots r_k$ s.t. $v = r_1 b_1 + r_2 b_2 + \ldots + r_k b_k$. Then,

$$\begin{aligned}
T(v) &= T(r_1 b_1 + r_2 b_2 + \ldots + r_k b_k) \\
&= T(r_1 b_1) + T(r_2 b_2) + \ldots + T(r_k b_k) \\
&= r_1 T(b_1) + r_2 T(b_2) + \ldots + r_k T(b_k) \\
&= r_1 T'(b_1) + r_2 T'(b_2) + \ldots + r_k T'(b_k) \\
&= T'(r_1 b_1) + T'(r_2 b_2) + \ldots + T'(r_k b_k) \\
&= T'(r_1 b_1 + r_2 b_2 + \ldots + r_k b_k) \\
&= T'(v)
\end{aligned}$$

$\therefore$ T and T' are the same linear transformation.

4. Perservation of Subspaces:

Let V and V' be subspaces.
Let $T: V \to V'$ be a linear transformation.
Then,

1. If W is a subspace of V, then $T[W]$ is a subspace of V'.

2. If W' is a subspace of V', then $T[W']$ is a subspace of V.

Proof:

1. Since $T[0] = 0'$, we only need to show that $T[W]$ is closed under addition and scalar multiplication.

   Let $T(w_1)$ and $T(w_2)$ be any vectors in $T(W)$ where w_1 and w_2 are vectors in W. Then, $T(w_1) + T(w_2) = T(w_1 + w_2)$ and $rT(w_1) = T(rw_1)$. Since w_1 and w_2 are in W, which is closed under addition, $T(w_1 + w_2)$ is in $T(W)$. This shows that $T(W)$ is closed under addition. Likewise, $rT(w_1)$ is in $T(W)$, so $T(W)$ is closed under scalar multiplication. $\therefore$ $T(W)$ is a subspace of V'.

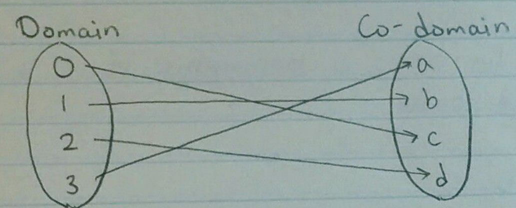
2. Since $0 \in T^{-1}(W')$, we only need to show that $T(W')$ is closed under addition and scalar multiplication. Let w_1' and w_2' be any two vectors in $T(W')$. Then,
   $T(w_1' + w_2') = T(w_1') + T(w_2')$
   $rT(w_1') = T(rw_1')$
   $\therefore$ $T^{-1}(W')$ is a subspace of V.

5. One-To-One, Onto, Isomorphic:

1. One-to-One (Injective):

A lin trans is one-to-one if each element in the domain maps to a distinct element in the co-domain. I.e. A lin trans is one-to-one iff its kernel = $\{0\}$.

E.g.

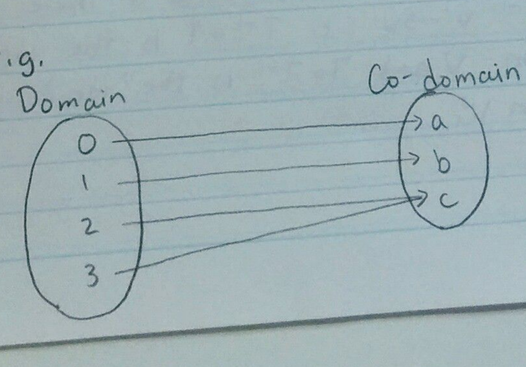


Here, each element of the domain maps to a unique element in the co-domain. I.e. No 2 elements in the domain map to the same value in the co-domain. Therefore, it is one-to-one.

2. Onto (Surjective):

A lin trans is onto (surjective) if every element in the co-domain is mapped onto by an element in the domain. I.e. If v' is in V', then there exists a v in V s.t. $T(v) = v'$. Note, the elements of the domain does not need to be unique.

E.g.



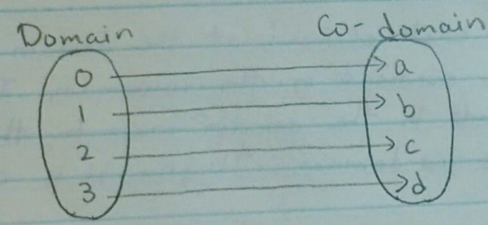
Since each element in the co-domain is mapped onto by an element in the domain, this is surjective. However, because both 2 and 3 in the domain map to c, this is not injective.

Note: A linear transformation is onto if the rank of the domain equal to the rank of the co-domain.

3. Isomorphism (Bijection):

A lin trans is isomorphic if it is both one-to-one and onto.

E.g.



This is both one-to-one and onto, so it's isomorphic.

6. Invertibility of a Linear Transformation:

A linear transformation is invertible iff it's isomorphic. I.e. Let V and V' be vector spaces. A lin trans is invertible if there exists a lin trans $T^{-1}: V' \to V$ s.t. $T^{-1} \circ T$ is the identity transformation on V and $T \circ T^{-1}$ is the identity transformation on V'.

6. Matrix Rep of Linear Transformations:

Let V and V' be finite-dimensional vector spaces and let $B = (b_1, b_2, \ldots b_n)$ and $B' = (b_1', \ldots b_m')$ be ordered basis for V and V' respectively. Let $T: V \to V'$ be a linear transformation, and let $T': \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation s.t. for each $v \in V$, we have $T'(v_B) = T(v)_{B'}$. Then, the standard matrix rep of T' is the matrix A whose j^{th} col vector is $T(b_j)_{B'}$ and $T(v)_{B'} = Av_B$ for all $v \in V$.

Let V and V' be n-dimensional vector spaces with ordered basis B and B' respectively. Let $T: V \to V'$ be an invertible linear transformation with a standard matrix rep of T relative to B, B'. Then, the standard matrix rep of T^{-1} relative to B', B is the inverse of the matrix rep of T relative to B, B'.

7. Solution Set of $T(x) = b$:

Let $T: V \to V'$ be a lin trans and let $T(p) = b$ for a particular vector p in V. The solution set of $T(x) = b$ is the set $\{p + h \mid h \in \ker(T)\}$.

E.g. Let V and V' be vector spaces having ordered basis $B = (b_1, b_2, b_3)$ and $B' = (b_1', b_2', b_3', b_4')$, respectively. Let $T: V \to V'$ be a linear transformation s.t.

$$T(b_1) = 3b_1' + b_2' + 4b_3' - b_4'$$
$$T(b_2) = b_1' + 2b_2' - b_3' + 2b_4'$$
$$T(b_3) = -2b_1' - b_2' + 2b_3'$$

Find the matrix rep A of T relative to B, B'.

Solution

$$A = \begin{bmatrix} | & | & & | \\ T(b_1)_{B'} & T(b_2)_{B'} & \cdots & T(b_n)_{B'} \\ | & | & & | \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & -1 \\ 4 & -1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

E.g. Let V and V' be vector spaces with ordered basis $B = (b_1, b_2, b_3)$ and $B' = (b_1', b_2', b_3', b_4')$, respectively. Let $T: V \to V'$ be a linear transformation having the given matrix A as a matrix rep relative to B, B'. Find $T(v)$ for the given vector v.

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} \qquad v = b_1 + b_2 + b_3$$

Solution:

$T(v)_{B'} = Av$

$$= \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 4 \\ 7 \\ 6 \end{bmatrix}$$

$T(v) = 4b_1' + 4b_2' + 7b_3' + 6b_4'$

Note: The matrix rep A of T relative to B, B' means that you are changing the base from B to B' and finding the matrix A relative to B'.

E.g. Let V and V' be subspaces with ordered basis $B = (b_1, b_2, b_3)$ and $B' = (b_1', b_2', b_3', b_4')$, respectively. Let $T: V \to V'$ be a linear transformation s.t.

$T(b_1) = b_1' + 2b_2' - 3b_3'$
$T(b_2) = 3b_1' + 5b_2' + 2b_3'$
$T(b_3) = -2b_1' - 3b_2' - 4b_3'$

a) Find the matrix A

$$A = \begin{bmatrix} | & | & & | \\ T(b_1)_{B'} & T(b_2)_{B'} & \dots & T(b_n)_{B'} \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix}$$

b) Use A to find $T(v)_{B'}$, if $v_B = [2, -5, 1]$

$$T(v)_{B'} = Av$$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 \\ -24 \\ -20 \end{bmatrix}$$

c) Show that T is invertible and find the matrix rep of T^{-1} relative to B', B.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -8 & -1 \\ 0 & 1 & 0 & -17 & 10 & 1 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right]$$

$$\therefore T^{-1}_{B',B} = A^{-1}$$

$$= \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix}$$

d) Find $T^{-1}(v')_B$ if $v'_B = [-1 \ 1 \ 3]$

$$T^{-1}(v')_B = A^{-1}(v'_B)$$

$$= \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -25 \\ 30 \\ 33 \end{bmatrix}$$